TURBULENCE MODELING FOR BEGINNERS

by

TONY SAAD

UNIVERSITY OF TENNESSEE SPACE INSTITUTE

tsaad@utsi.edu – http://jedi.knows.it

The purpose of this tiny guide is to summarize the basic concepts of turbulence modeling and to a compile the fundamental turbulence models into one simple framework. Intended for the beginner, no derivations are included, unless in some simple cases, as the focus is to present a balance between the physical understanding and the closure equations. I hope this material will be helpful.

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INTRODUCTION

A turbulent flow field is characterized by velocity fluctuations in all directions and has an infinite number of scales (degrees of freedom). Solving the NS equations for a turbulent flow is impossible because the equations are elliptic, non-linear, coupled (pressure-velocity, temperature-velocity). The flow is three dimensional, chaotic, diffusive, dissipative, and intermittent. The most important characteristic of a turbulent flow is the infinite number of scales so that a full numerical resolution of the flow requires the construction of a grid with a number of nodes that is proportional to $Re^{9/4}$.

The governing equations for a Newtonian fluid are

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho \tilde{u}_i}{\partial x_i} = 0 \tag{1}$$

Conservation of momentum

$$\frac{\partial \rho \tilde{u}_i}{\partial \tau} + \frac{\partial \rho \tilde{u}_j \tilde{u}_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \tilde{u}_i}{\partial x_j}\right) - \frac{\partial \tilde{p}}{\partial x_i} + \rho g_i + \tilde{s}_{ui} \tag{2}$$

Conservation of passive scalars (given a scalar $\tilde{T}$)

$$\frac{\partial \rho c_p \tilde{T}}{\partial \tau} + \frac{\partial \rho c_p \tilde{u}_j \tilde{T}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k \frac{\partial \tilde{T}}{\partial x_j}\right) + \tilde{s}_t \tag{3}$$

So how can we solve the problem? One of the solutions is to reduce the number of scales (from infinity to 1 or 2) by using the Reynolds decomposition. Any property (whether a vector or a scalar) can be written as the sum of an average and a fluctuation, i.e. $\tilde{\phi} = \Phi + \phi$ where the capital letter denotes the average and the lower case letter denotes the fluctuation of the property. Of course, this decomposition will yield a set of equations governing the average flow field. The new equations will be exact for an average flow field not for the exact turbulent flow field. By an average flow field we mean that any property becomes constant over

time. The result of using the Reynolds decomposition in the NS equations is called the RANS or Reynolds Averaged Navier Stokes Equations. Upon substitution of the Reynolds decomposition (for each variable, we substitute the corresponding decomposition) we obtain the following RANS equations:

Conservation of Mass

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial \rho U_i}{\partial x_i} = 0 \tag{4}$$

Conservation of momentum

$$\frac{\partial \rho U_i}{\partial \tau} + \frac{\partial \rho U_j U_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\mu \frac{\partial U_i}{\partial x_j} - \rho \overline{u_i u_j}\right) - \frac{\partial P}{\partial x_i} + S_{ui} \tag{5}$$

Conservation of passive scalars (given a scalar $\tilde{T}$)

$$\frac{\partial \rho c_p T}{\partial \tau} + \frac{\partial \rho c_p U_j T}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j} - \rho c_p \overline{u_j t}\right) + S_t \tag{6}$$

Note: a special property of the Reynolds decomposition is that the average of the fluctuating component is identically zero, a fact that is used in the derivation of the above equations.
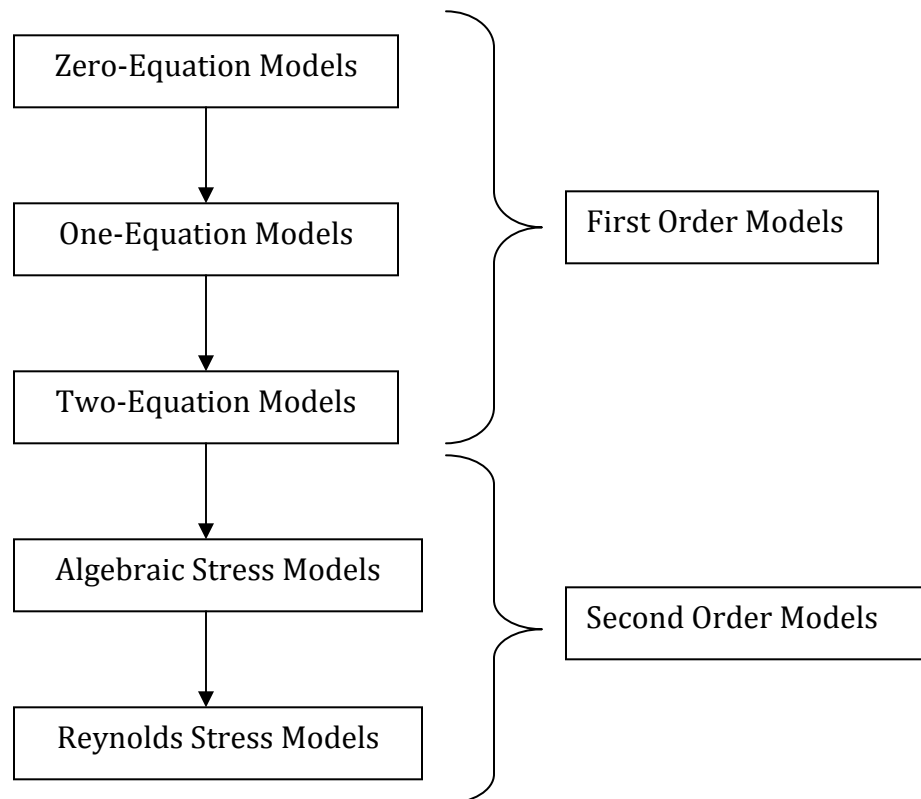
However, by using the Reynolds decomposition, there are new unknowns that were introduced such as the turbulent stresses $\rho \overline{u_i u_j}$ and turbulent fluxes (where the overbar denotes an average) and therefore, the RANS equations describe an open set of equations. The need for additional equations to model the new unknowns is called *Turbulence Modeling*.

We now have 9 additional unknowns (6 Reynolds stresses and 3 turbulent fluxes). In total, for the simplest turbulent flow (including the transport of a scalar passive scalar, e.g. temperature when heat transfer is involved) there 14 unknowns!

A straight forward method to model the additional unknowns is to develop new PDEs for each term by using the original set of the NS equations (multiplying the momentum equations to produce the turbulent stresses...). However, the problem with this procedure is that it will introduce new correlations for the unknowns

(triple correlations) and so on. We then might think of developing new equations for the triple correlations, nevertheless, we will end up with quadruple correlations... and so on... An alternative approach is to use the PDEs for the turbulent stresses and fluxes as a guide to modeling. The turbulent models are as follows, in order of increasing complexity:

- Algebraic (zero equation) models: mixing length (first order model)

- One equation models: k-model, μ_t-model (first order model)

- Two equation models: k-ε, k-kl, k-ω^2, low Re k-ε (first order model)

- Algebraic stress models: ASM (second order model)

- Reynolds stress models: RSM (second order model)

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FIRST ORDER MODELS

First order models are based on the analogy between laminar and turbulent flow. They are also called Eddy Viscosity Models (EVM). The idea is that the average turbulent flow field is similar to the corresponding laminar flow. This analogy is illustrated as follows

$$\text{Laminar FLow:} \begin{cases} \tau_{ij} = \mu\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) - \dfrac{2}{3}\mu\delta_{ij}\dfrac{\partial u_j}{\partial x_j} \\[4mm] q_i = \dfrac{k}{c_p}\dfrac{\partial T}{\partial x_i} \end{cases}$$

$$\tag{7}$$

$$\text{Turbulent Flow:} \begin{cases} \tau_{ij}^{t} = \rho\overline{u_i u_j} = \mu_t\left(\dfrac{\partial U_i}{\partial x_j} + \dfrac{\partial U_j}{\partial x_i}\right) - \dfrac{2}{3}\delta_{ij}\rho k \\[4mm] q_i^{t} = -\rho\overline{u_i t} = \dfrac{k_t}{c_p}\dfrac{\partial T}{\partial x_i} \end{cases}$$

which is referred to as the generalized Boussinesq hypothesis.

Note that:

$\mu_t = $ Turbulent Viscosity
$k = $ Turbulent Kinetic Energy
$k_t = $ Turbulent Conduction Coefficient

The turbulent viscosity and the turbulent conduction coefficient are flow properties. They are not properties of the fluid. They vary from one flow to another. So the problem now is the devise means or models to find these unknowns, the turbulent viscosity and the turbulent conduction, because the turbulent stresses and fluxes will be expressed as function of these new flow properties.

In zero equation models, as the name designates, we have no PDE that describes the transport of the turbulent stresses and fluxes. A simple algebraic relation is used to close the problem. Based on the mixing length theory, which is the length over which there is high interaction of vortices in a turbulent flow field, dimensional analysis is used to show that:

$$v_t = \frac{\mu_t}{\rho} \sim lu = l_m \left(l_m \frac{dU}{dy} \right) \tag{8}$$

l_m is determined experimentally. For boundary layers, we have

$$\begin{aligned} l_m &= \kappa y \text{ for } y < \delta \\ l_m &= \delta \text{ for } y \geq \delta \end{aligned} \tag{9}$$

and

$$k_t = \mu_t \frac{c_p}{\text{Pr}_t} \tag{10}$$

Equations (8) through (10) are then used in the laminar-turbulent analogy and then back into the original RANS equations.

In one-equation models, a PDE is derived for the turbulent kinetic energy and the unknowns (turbulent viscosity and conduction coefficient) are expressed as a function of the turbulent kinetic energy as:

$$K = \frac{1}{2}\left(\overline{u^2} + \overline{v^2} + \overline{w^2}\right) \tag{11}$$

We also make use of the fact that $\mu_t \propto lu$ but in this case, the velocity scales is proportional to the square root of the kinetic energy (unlike the above case where u was proportional to the gradient of velocity). Therefore, we have:

$$\mu_t = C_\mu \sqrt{K}\, l_m$$

$$k_t = \mu_t \frac{c_p}{\mathrm{Pr}_t} \tag{12}$$

Now that the turbulent viscosity and turbulent conduction coefficient are expressed in terms of the turbulent kinetic energy (therefore the turbulent stresses and turbulent fluxes are functions of the kinetic energy), a PDE is developed for the turbulent kinetic energy.

$$\frac{\partial \rho K}{\partial \tau} + \frac{\partial \rho U_j K}{\partial x_j} = -\rho\overline{u_i u_j}\frac{\partial \rho U_i}{\partial x_j} - \beta g_i \rho \overline{u_i t} \; ....(\mathrm{P_k} + G_k)$$

$$-\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho\overline{u_i^2 u_j} + \overline{pu_j} - \mu\frac{\partial K}{\partial x_j}\right) \; ....(\mathrm{D_k}) \tag{13}$$

$$-\mu\frac{\overline{\partial U_j}}{\partial x_j}\frac{\partial U_j}{\partial x_j} \; ....(\varepsilon_\mathrm{k})$$

On the other hand, this equation introduces two new unknown correlations; the turbulent and pressure diffusions (D_k) and the dissipation rates (ε_k) which need to be modeled. Finally, we end up with the following:

$$\frac{\partial \rho K}{\partial \tau} + \frac{\partial \rho U_j K}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial K}{\partial x_j}\right] + P_k + G_k - \rho\varepsilon \tag{14}$$

Where

$$P_k = \rho \overline{u_i u_j} \frac{\partial \rho U_i}{\partial x_j} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} \tag{15}$$

$$G_k = \beta g_i \frac{k_t}{c_p} \frac{\partial T}{\partial x_i} \tag{16}$$

And from dimensional analysis, we obtain

$$\varepsilon = \mu \overline{\frac{\partial U_j}{\partial x_j} \frac{\partial U_j}{\partial x_j}} \propto \frac{K^{3/2}}{l} \Rightarrow \varepsilon = C_d \frac{K^{3/2}}{l_m} \tag{17}$$

In the two-equation models, we develop two PDEs: one for the turbulent kinetic energy and one for the turbulent dissipation rate. The PDE for the turbulent kinetic energy is already given by Eq.(13), however, the expression for the turbulent or eddy viscosity is different. So, the idea is to express the turbulent viscosity as a function of K and ε and then derive PDEs for K and ε.

$$\mu_t \propto ul = k^{1/2} \left(\frac{k^{3/2}}{\varepsilon} \right)$$

$$\mu_t = C_\mu \frac{k^2}{\varepsilon} \tag{18}$$

$$k_t = \mu_t \frac{c_p}{\text{Pr}_t}$$

The equation for the turbulent kinetic energy is repeated here for convenience

$$\frac{\partial \rho K}{\partial \tau} + \frac{\partial \rho U_j K}{\partial x_j} = \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial K}{\partial x_j} \right] + P_k + G_k - \rho \varepsilon \tag{19}$$

$$P_k = \rho \overline{u_i u_j} \frac{\partial \rho U_i}{\partial x_j} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} \tag{20}$$

$$G_k = \beta g_i \frac{k_t}{c_p} \frac{\partial T}{\partial x_i} \tag{21}$$

Now instead of modeling ε, we shall develop an independent PDE for its transport. We obtain

$$\frac{\partial \rho \varepsilon}{\partial \tau} + \frac{\partial \rho U_j \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right] + C_{\varepsilon 1} \left(P_k + C_{\varepsilon 3} G_k \right) \frac{\varepsilon}{K} - C_{\varepsilon 2} \rho \frac{\varepsilon^2}{K} \tag{22}$$

$$P_k = \rho \overline{u_i u_j} \frac{\partial \rho U_i}{\partial x_j} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} \tag{23}$$

$$G_k = \beta g_i \frac{k_t}{c_p} \frac{\partial T}{\partial x_i} \qquad (24)$$

The constants are determined from simple benchmark experiments.

C_μ	σ_k	σ_ε	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	$C_{\varepsilon 3}$	Pr_t
0.09	1.0	1.3	1.44	1.92	0 - 1.0	0.7 – 0.9

SECOND ORDER MODELS

The central concept of second order models is to make direct use of the governing equations for the second order moments (Reynolds stresses and turbulent fluxes) instead of the questionable Boussinesq hypothesis. The motivation is to overcome the limitations of first order models in dealing with the isotropy of turbulence and the extra strains. The overshoot of this approach is the large number of PDEs induced which involve many unknown or impossible to find correlations. The most famous models are the Algebraic Stress Model (ASM) and the Reynolds Stress Model (RSM). The second order modeling approach shall be illustrated with the RSM model only.

The RSM involves the modeling of turbulent diffusion, pressure strain correlation which is the most involved part of the RSM, and the turbulent dissipation rate. The RANS momentum equation is written as

$$\frac{\partial \rho \overline{u_i u_j}}{\partial \tau} + \frac{\partial \rho U_k \overline{u_i u_j}}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\mu \frac{\partial \overline{u_i u_j}}{\partial x_k} + D_{ij}^t\right)$$
$$+ P_{ij} + G_{ij} + \Phi_{ij} - \varepsilon_{ij}$$

(25)

Where

$$P_{ij} = -\rho \overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \rho \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}$$

(26)

$$G_{ij} = -\rho \beta g_j \overline{u_i t} - \rho \beta g_i \overline{u_j t}$$

(27)

And the scalar transport equation is

$$\frac{\partial \rho \overline{u_i t}}{\partial \tau} + \frac{\partial \rho U_j \overline{u_i t}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\alpha \frac{\partial \overline{u_i t}}{\partial x_j} + D_{it}^t\right)$$
$$+ P_{it} + G_{it} + \Phi_{it} - \varepsilon_{it}$$

(28)

Where

$$P_{it} = P_{it1} + P_{it2} = -\rho \overline{u_i u_k}\frac{\partial T}{\partial x_k} - \rho \overline{u_k t}\frac{\partial U_i}{\partial x_k}$$

(29)

$$G_{it} = -\rho \beta g_i \overline{t^2}$$

(30)

The objective is to find models for turbulent diffusion (D_{ij}^t, D_{it}^t), the pressure-strain correlation (Φ_{ij}, Φ_{it}), and the turbulent dissipation rate ($\varepsilon_{ij}, \varepsilon_{it}$).

TURBULENT DIFFUSION MODELING

One way to model the 3rd order turbulent diffusion tensor is to write its own transport equation. However, this becomes very complicated (handling tensors by

itself is sometimes rather involved!). One remedy is to use Generalized Gradient Diffusion Hypothesis

$$D_{ij}^t = C_s \frac{k}{\varepsilon} \left(\overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l} \right) \tag{31}$$

and

$$D_{it}^t = C_{st} \frac{k}{\varepsilon} \left(\overline{u_i u_k} \frac{\partial \overline{u_j t}}{\partial x_k} + \overline{u_j u_k} \frac{\partial \overline{u_i t}}{\partial x_k} \right) \tag{32}$$

where the constants are determined from simple experiments.

PRESSURE-STRAIN CORRELATION MODELING

The role of the pressure-strain interaction is to redistribute the turbulent kinetic energy over the three normal stresses. The governing equation for the evolution of this phenomenon takes the form of a Lagrangian integro-differential equation

$$\overline{\frac{p}{\rho} \frac{\partial u_i}{\partial x_j}} = \frac{1}{4\pi} \int_V \left[\overline{\left(\frac{\partial^2 u_k' u_l'}{\partial x_k' \partial x_l'} \right) \frac{\partial u_i}{\partial x_j}} + 2 \frac{\partial U_k'}{\partial x_l'} \overline{\frac{\partial u_l'}{\partial x_k'} \frac{\partial u_i}{\partial x_j}} + \frac{\alpha g_k}{T'} \overline{\frac{\partial t'}{\partial x_k'} \frac{\partial u_i}{\partial x_j}} \right] \frac{dV}{x - x'} + I_S \tag{33}$$

With a similar equation for pressure-scalar correlation. Without further complication, the final analysis assumes the following

$$\Phi_{ij} = \Phi_{ij1} + \Phi_{ij2} + \Phi_{ij3} + \Phi_{ij1}^w + \Phi_{ij2}^w + \Phi_{ij3}^w \tag{34}$$

And

$$\Phi_{it} = \Phi_{it1} + \Phi_{it2} + \Phi_{it3} + \Phi_{it1}^w + \Phi_{it2}^w + \Phi_{it3}^w \tag{35}$$

where

Φ_{ij1}, Φ_{it1} represent the turbulence-turbulence interaction, i.e. between fluctuations. They are modeled using an isotropic assumption, as in the decay of homogeneous turbulence

$$\Phi_{ij1} = -C_1 \rho \varepsilon a_{ij} = -C_1 \frac{\varepsilon}{k}\left(\overline{\rho u_i u_j} - \frac{2}{3}\delta_{ij}\rho k \right) \tag{36}$$

$$\Phi_{it1} = -C_{1t}\overline{\rho u_i t}\frac{1}{\tau_9} = -C_{1t}\overline{\rho u_i t}\frac{\varepsilon}{k} \tag{37}$$

Φ_{ij2}, Φ_{it2} represent the shear effects. They are also modeled by "isotropization" of turbulence

$$\Phi_{ij2} = -C_2\left(P_{ij} - \frac{1}{3}\delta_{ij}P_{kk} \right) \tag{38}$$

$$\Phi_{it2} = -C_{2t}P_{it2} \tag{39}$$

Φ_{ij3}, Φ_{it3} represent the body force effects. Their modeling is similar to that of the shear effects

$$\Phi_{ij3} = -C_3\left(G_{ij} - \frac{1}{3}\delta_{ij}G_{kk} \right) \tag{40}$$

$$\Phi_{it3} = -C_{3t}G_{it} \tag{41}$$

Finally, $\Phi_{ij1}^{w},\Phi_{it1}^{w},\Phi_{ij2}^{w},\Phi_{it2}^{w},\Phi_{ij3}^{w},\Phi_{it3}^{w}$ represent the wall effects. The basic idea is that a pressure wave is reflected at a wall and thus affects the whole flow field in an elliptic manner

$$\Phi_{ij1}^{w} = C_{1w}\rho\frac{\varepsilon}{k}\left(\overline{u_k u_l}n_k n_l \delta_{ij} - \frac{3}{2}\overline{u_i u_k}n_j n_k - \frac{3}{2}\overline{u_j u_k}n_i n_k \right)\mathsf{F}_n$$

$$\Phi_{ij2}^{w} = C_{2w}\left(\Phi_{kl2}n_k n_l \delta_{ij} - \frac{3}{2}\Phi_{ik2}n_j n_k - \frac{3}{2}\Phi_{jk2}n_i n_k \right)\mathsf{F}_n \tag{42}$$

$$\Phi_{ij3}^{w} = C_{3w}\left(\Phi_{kl3}n_k n_l \delta_{ij} - \frac{3}{2}\Phi_{ik3}n_j n_k - \frac{3}{2}\Phi_{jk3}n_i n_k \right)\mathsf{F}_n$$

and

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$$\Phi_{it1}^w = -C_{1tw}\rho\frac{\varepsilon}{k}\overline{u_k t}\, n_i n_k \mathsf{F}_n$$

$$\Phi_{it2}^w = C_{2tw}\Phi_{kt2}n_i n_k \mathsf{F}_n \tag{43}$$

$$\Phi_{it3}^w = C_{3tw}\Phi_{kt3}n_i n_k \mathsf{F}_n$$

with

$$\mathsf{F}_n = \frac{C_\mu^{3/4}}{\kappa}\frac{K^{3/2}}{\varepsilon y_n} \tag{44}$$

Where y_n represents the distance from the wall to the concerned point in the domain.

MODELING OF THE TURBULENT DISSIPATION RATE

A simple isotropic model is used to model ε.

$$\varepsilon_{ij} = \frac{2}{3}\delta_{ij}\varepsilon \tag{45}$$

$$\varepsilon_{it} = 0 \tag{46}$$

With the following transport equation for the dissipation rate

$$\frac{\partial\rho\varepsilon}{\partial\tau} + \frac{\partial\rho U_j\varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left[C_\varepsilon\rho\overline{u_j u_k}\frac{k}{\varepsilon}\frac{\partial\varepsilon}{\partial x_k}\right] + \frac{\varepsilon}{k}\left(C_{\varepsilon1}P_k - C_{\varepsilon2}\rho\varepsilon\right)P_k = \frac{1}{2}P_{ii} = -\rho\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} \tag{47}$$

Where

$$P_k = \frac{1}{2}P_{ii} = -\rho\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} \tag{48}$$

The constants appearing in RSM equations are given in the following table

C_s	C_1	C_2	C_3	C_{1w}	C_{2w}	C_{3w}	C_ε	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	C_t	C_{1t}	C_{2t}	C_{3t}
0.22	1.8	0.6	0.5	0.5	0.3	0.3	0.18	1.44	1.92	0.2	3.0	0.5	0.5

C_{1tw}	C_{2tw}	C_{3tw}
0.5	0.0	0.0

In the algebraic stress model, two main approaches can be undertaken. In the first, the transport of the turbulent stresses is assumed proportional to the turbulent kinetic energy; while in the second, convective and diffusive effects are assumed to be negligible. Algebraic Stress models can only be used where convective and diffusive fluxes are negligible, i.e. source dominated flows.

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